

5.7 Inverse Trigonometric Functions: Integration

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you (see Exercises 75–77).

■ FOR FURTHER INFORMATION

For a detailed proof of rule 2 of Theorem 5.17, see the article “A Direct Proof of the Integral Formula for Arctangent” by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to MathArticles.com.

THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x , and let $a > 0$.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

EXAMPLE 1 Integration with Inverse Trigonometric Functions

- a. $\int \frac{dx}{\sqrt{4-x^2}} = \arcsin \frac{x}{2} + C$
- b. $\int \frac{dx}{2+9x^2} = \frac{1}{3} \int \frac{3 dx}{(\sqrt{2})^2 + (3x)^2} \quad u = 3x, a = \sqrt{2}$
 $= \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$
- c. $\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2 dx}{2x\sqrt{(2x)^2-3^2}} \quad u = 2x, a = 3$
 $= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$

The integrals in Example 1 are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

EXAMPLE 2 Integration by Substitution

$$\text{Find } \int \frac{dx}{\sqrt{e^{2x} - 1}}.$$

Solution As it stands, this integral doesn't fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces

$$u = e^x \quad \Rightarrow \quad du = e^x dx \quad \Rightarrow \quad dx = \frac{du}{e^x} = \frac{du}{u}.$$

With this substitution, you can integrate as shown.

$$\begin{aligned} \int \frac{dx}{\sqrt{e^{2x} - 1}} &= \int \frac{dx}{\sqrt{(e^x)^2 - 1}} && \text{Write } e^{2x} \text{ as } (e^x)^2. \\ &= \int \frac{du/u}{\sqrt{u^2 - 1}} && \text{Substitute.} \\ &= \int \frac{du}{u\sqrt{u^2 - 1}} && \text{Rewrite to fit Arcsecant Rule.} \\ &= \operatorname{arcsec} \frac{|u|}{1} + C && \text{Apply Arcsecant Rule.} \\ &= \operatorname{arcsec} e^x + C && \text{Back-substitute.} \end{aligned}$$

▷ **TECHNOLOGY PITFALL** A symbolic integration utility can be useful for integrating functions such as the one in Example 2. In some cases, however, the utility may fail to find an antiderivative for two reasons. First, some elementary functions do not have antiderivatives that are elementary functions. Second, every utility has limitations—you might have entered a function that the utility was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x} - 1}} = \arctan \sqrt{e^{2x} - 1} + C.$$

Try showing that this antiderivative is equivalent to the one found in Example 2.

EXAMPLE 3 Rewriting as the Sum of Two Quotients

$$\text{Find } \int \frac{x + 2}{\sqrt{4 - x^2}} dx.$$

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned} \int \frac{x + 2}{\sqrt{4 - x^2}} dx &= \int \frac{x}{\sqrt{4 - x^2}} dx + \int \frac{2}{\sqrt{4 - x^2}} dx \\ &= -\frac{1}{2} \int (4 - x^2)^{-1/2} (-2x) dx + 2 \int \frac{1}{\sqrt{4 - x^2}} dx \\ &= -\frac{1}{2} \left[\frac{(4 - x^2)^{1/2}}{1/2} \right] + 2 \arcsin \frac{x}{2} + C \\ &= -\sqrt{4 - x^2} + 2 \arcsin \frac{x}{2} + C \end{aligned}$$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

EXAMPLE 4 Completing the Square

•••▶ See LarsonCalculus.com for an interactive version of this type of example.

Find $\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares, as shown.

$$x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7 = (x - 2)^2 + 3 = u^2 + a^2$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

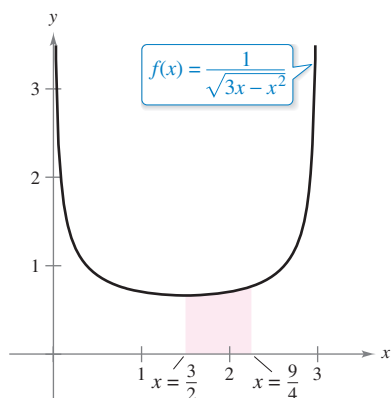
$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x - 2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x - 2}{\sqrt{3}} + C$$

When the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square of $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x - 2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$3x - x^2 = -(x^2 - 3x) = -\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] = \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$$



The area of the region bounded by the graph of f , the x -axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.

Figure 5.28

- ▶ **TECHNOLOGY** With definite integrals such as the one given in Example 5, remember that you can resort to a numerical solution. For instance, applying Simpson's Rule (with $n = 12$) to the integral in the example, you obtain
- $$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} dx \approx 0.523599.$$
- This differs from the exact value of the integral ($\pi/6 \approx 0.5235988$) by less than one-millionth.

EXAMPLE 5 Completing the Square

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x -axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution In Figure 5.28, you can see that the area is

$$\begin{aligned} \text{Area} &= \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}} dx \\ &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \\ &= \arcsin \frac{x - (3/2)}{3/2} \Big|_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524. \end{aligned}$$

Use completed square form derived above.

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules**. To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u) du = k \int f(u) du$
2. $\int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$
3. $\int du = u + C$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u du = e^u + C$
7. $\int a^u du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u du = -\cos u + C$
9. $\int \cos u du = \sin u + C$
10. $\int \tan u du = -\ln|\cos u| + C$
11. $\int \cot u du = \ln|\sin u| + C$
12. $\int \sec u du = \ln|\sec u + \tan u| + C$
13. $\int \csc u du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u du = \tan u + C$
15. $\int \csc^2 u du = -\cot u + C$
16. $\int \sec u \tan u du = \sec u + C$
17. $\int \csc u \cot u du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on during the development of differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More important, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* solve with the techniques and rules you now know.

EXAMPLE 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{dx}{x\sqrt{x^2 - 1}}$

b. $\int \frac{x dx}{\sqrt{x^2 - 1}}$

c. $\int \frac{dx}{\sqrt{x^2 - 1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned} \int \frac{x dx}{\sqrt{x^2 - 1}} &= \frac{1}{2} \int (x^2 - 1)^{-1/2} (2x) dx \\ &= \frac{1}{2} \left[\frac{(x^2 - 1)^{1/2}}{1/2} \right] + C \\ &= \sqrt{x^2 - 1} + C \end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. (You should scan the list of basic integration rules to verify this conclusion.)

EXAMPLE 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{dx}{x \ln x}$

b. $\int \frac{\ln x dx}{x}$

c. $\int \ln x dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned} \int \frac{dx}{x \ln x} &= \int \frac{1/x}{\ln x} dx \\ &= \ln|\ln x| + C \end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned} \int \frac{\ln x dx}{x} &= \int \left(\frac{1}{x}\right) (\ln x)^1 dx \\ &= \frac{(\ln x)^2}{2} + C \end{aligned}$$

•• **REMARK** Note in
 • Examples 6 and 7 that the
 • *simplest* functions are the ones
 • that you cannot yet integrate.
 •••••

c. You *cannot* find this integral using the techniques you have studied so far.

5.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.**Finding an Indefinite Integral** In Exercises 1–20, find the indefinite integral.

1. $\int \frac{dx}{\sqrt{9-x^2}}$
2. $\int \frac{dx}{\sqrt{1-4x^2}}$
3. $\int \frac{1}{x\sqrt{4x^2-1}} dx$
4. $\int \frac{12}{1+9x^2} dx$
5. $\int \frac{1}{\sqrt{1-(x+1)^2}} dx$
6. $\int \frac{1}{4+(x-3)^2} dx$
7. $\int \frac{t}{\sqrt{1-t^4}} dt$
8. $\int \frac{1}{x\sqrt{x^4-4}} dx$
9. $\int \frac{t}{t^4+25} dt$
10. $\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx$
11. $\int \frac{e^{2x}}{4+e^{4x}} dx$
12. $\int \frac{2}{x\sqrt{9x^2-25}} dx$
13. $\int \frac{\sec^2 x}{\sqrt{25-\tan^2 x}} dx$
14. $\int \frac{\sin x}{7+\cos^2 x} dx$
15. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$
16. $\int \frac{3}{2\sqrt{x}(1+x)} dx$
17. $\int \frac{x-3}{x^2+1} dx$
18. $\int \frac{x^2+3}{x\sqrt{x^2-4}} dx$
19. $\int \frac{x+5}{\sqrt{9-(x-3)^2}} dx$
20. $\int \frac{x-2}{(x+1)^2+4} dx$

Evaluating a Definite Integral In Exercises 21–32, evaluate the definite integral.

21. $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}} dx$
22. $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}} dx$
23. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2} dx$
24. $\int_{\sqrt{3}}^3 \frac{1}{x\sqrt{4x^2-9}} dx$
25. $\int_3^6 \frac{1}{25+(x-3)^2} dx$
26. $\int_1^4 \frac{1}{x\sqrt{16x^2-5}} dx$
27. $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}} dx$
28. $\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx$
29. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$
30. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$
31. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx$
32. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}} dx$

Completing the Square In Exercises 33–42, find or evaluate the integral by completing the square.

33. $\int_0^2 \frac{dx}{x^2-2x+2}$
34. $\int_{-2}^2 \frac{dx}{x^2+4x+13}$
35. $\int \frac{2x}{x^2+6x+13} dx$
36. $\int \frac{2x-5}{x^2+2x+2} dx$
37. $\int \frac{1}{\sqrt{-x^2-4x}} dx$
38. $\int \frac{2}{\sqrt{-x^2+4x}} dx$

39. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx$
40. $\int \frac{1}{(x-1)\sqrt{x^2-2x}} dx$
41. $\int \frac{x}{x^4+2x^2+2} dx$
42. $\int \frac{x}{\sqrt{9+8x^2-x^4}} dx$

Integration by Substitution In Exercises 43–46, use the specified substitution to find or evaluate the integral.

43. $\int \sqrt{e^t-3} dt$
 $u = \sqrt{e^t-3}$
44. $\int \frac{\sqrt{x-2}}{x+1} dx$
 $u = \sqrt{x-2}$
45. $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$
 $u = \sqrt{x}$
46. $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$
 $u = \sqrt{x+1}$

WRITING ABOUT CONCEPTS**Comparing Integration Problems** In Exercises 47–50, determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

47. (a) $\int \frac{1}{\sqrt{1-x^2}} dx$
48. (a) $\int e^{x^2} dx$
- (b) $\int \frac{x}{\sqrt{1-x^2}} dx$
- (b) $\int xe^{x^2} dx$
- (c) $\int \frac{1}{x\sqrt{1-x^2}} dx$
- (c) $\int \frac{1}{x^2} e^{1/x} dx$
49. (a) $\int \sqrt{x-1} dx$
50. (a) $\int \frac{1}{1+x^4} dx$
- (b) $\int x\sqrt{x-1} dx$
- (b) $\int \frac{x}{1+x^4} dx$
- (c) $\int \frac{x}{\sqrt{x-1}} dx$
- (c) $\int \frac{x^3}{1+x^4} dx$

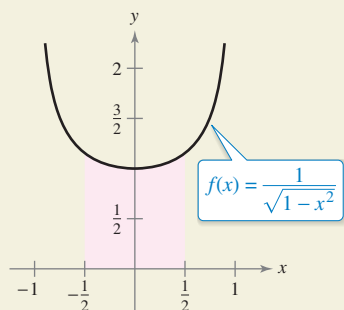
51. Finding an Integral Decide whether you can find the integral

$$\int \frac{2 dx}{\sqrt{x^2+4}}$$

using the formulas and techniques you have studied so far. Explain your reasoning.



52. HOW DO YOU SEE IT? Using the graph, which value best approximates the area of the region between the x -axis and the function over the interval $[-\frac{1}{2}, \frac{1}{2}]$? Explain.



- (a) -3 (b) $\frac{1}{2}$ (c) 1 (d) 2 (e) 4

Differential Equation In Exercises 53 and 54, use the differential equation and the specified initial condition to find y .

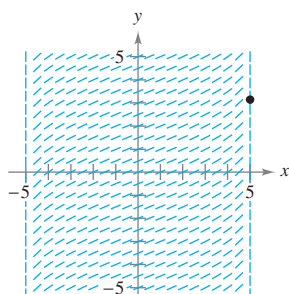
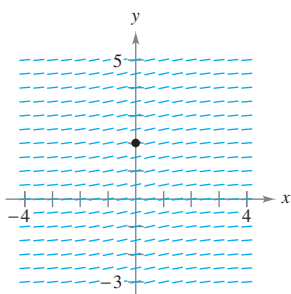
53. $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$
 $y(0) = \pi$

54. $\frac{dy}{dx} = \frac{1}{4+x^2}$
 $y(2) = \pi$

Graphing Utility In Exercises 55 and 56, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to MathGraphs.com.

55. $\frac{dy}{dx} = \frac{2}{9+x^2}$, $(0, 2)$

56. $\frac{dy}{dx} = \frac{2}{\sqrt{25-x^2}}$, $(5, \pi)$



Graphing Utility In Exercises 57–60, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.

57. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$
 $y(3) = 0$

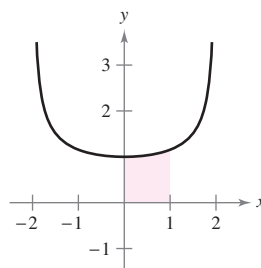
58. $\frac{dy}{dx} = \frac{1}{12+x^2}$
 $y(4) = 2$

59. $\frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}$
 $y(0) = 2$

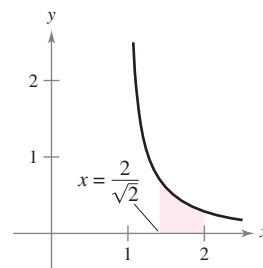
60. $\frac{dy}{dx} = \frac{\sqrt{y}}{1+x^2}$
 $y(0) = 4$

Area In Exercises 61–66, find the area of the region.

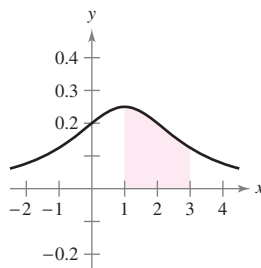
61. $y = \frac{2}{\sqrt{4-x^2}}$



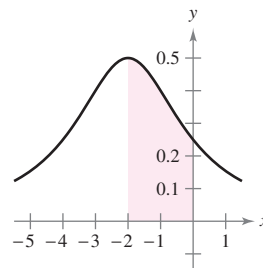
62. $y = \frac{1}{x\sqrt{x^2-1}}$



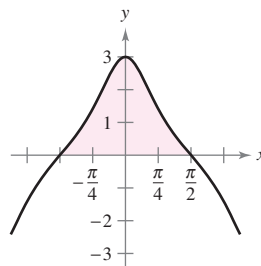
63. $y = \frac{1}{x^2-2x+5}$



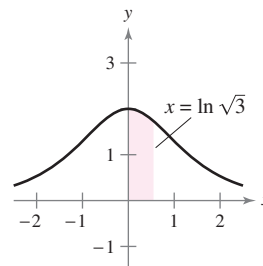
64. $y = \frac{2}{x^2+4x+8}$



65. $y = \frac{3 \cos x}{1 + \sin^2 x}$



66. $y = \frac{4e^x}{1 + e^{2x}}$



67. Area

(a) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x \, dx.$$

Graphing Utility (b) Use the integration capabilities of a graphing utility to approximate the area.

(c) Find the exact area analytically.

68. Approximating Pi

(a) Show that

$$\int_0^1 \frac{4}{1+x^2} \, dx = \pi.$$

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral in part (a).

Graphing Utility (c) Approximate the number π by using the integration capabilities of a graphing utility.

69. Investigation Consider the function

$$F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2 + 1} dt.$$

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function

$$f(x) = \frac{2}{x^2 + 1}.$$

Use what you have written to guess the value of x that will make F maximum.


(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

70. Comparing Integrals Consider the integral

$$\int \frac{1}{\sqrt{6x - x^2}} dx.$$

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

 (c) The antiderivatives in parts (a) and (b) appear to be significantly different. Use a graphing utility to graph each antiderivative in the same viewing window and determine the relationship between them. Find the domain of each.

True or False? In Exercises 71–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

71. $\int \frac{dx}{3x\sqrt{9x^2 - 16}} = \frac{1}{4} \operatorname{arcsec} \frac{3x}{4} + C$

72. $\int \frac{dx}{25 + x^2} = \frac{1}{25} \arctan \frac{x}{25} + C$

73. $\int \frac{dx}{\sqrt{4 - x^2}} = -\arccos \frac{x}{2} + C$

74. One way to find $\int \frac{2e^{2x}}{\sqrt{9 - e^{2x}}} dx$ is to use the Arcsine Rule.

Verifying an Integration Rule In Exercises 75–77, verify the rule by differentiating. Let $a > 0$.

75. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

76. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

77. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

78. Proof Graph

$$y_1 = \frac{x}{1 + x^2}, \quad y_2 = \arctan x, \quad \text{and} \quad y_3 = x$$

on $[0, 10]$. Prove that

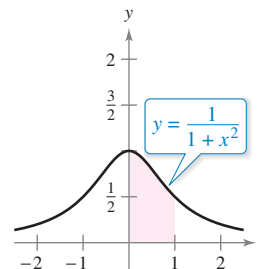
$$\frac{x}{1 + x^2} < \arctan x < x \quad \text{for} \quad x > 0.$$


79. Numerical Integration

(a) Write an integral that represents the area of the region in the figure.

(b) Use the Trapezoidal Rule with $n = 8$ to estimate the area of the region.

(c) Explain how you can use the results of parts (a) and (b) to estimate π .



 **80. Vertical Motion** An object is projected upward from ground level with an initial velocity of 500 feet per second. In this exercise, the goal is to analyze the motion of the object during its upward flight.

(a) If air resistance is neglected, find the velocity of the object as a function of time. Use a graphing utility to graph this function.

(b) Use the result of part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where -32 feet per second per second is the acceleration due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32 + kv^2} = - \int dt.$$


(d) Use a graphing utility to graph the velocity function $v(t)$ in part (c) for $k = 0.001$. Use the graph to approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t) dt$$

where $v(t)$ and t_0 are those found in part (d). This is the approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) and (e).

 **FOR FURTHER INFORMATION** For more information on this topic, see the article “What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?” by John Lekner in *Mathematics Magazine*. To view this article, go to MathArticles.com.